SIMILARITY OF NON-NEWTONIAN FLOWS. I. RHEOLOGICAL SIMILARITY

O. WEIN[a], K. WICHTERLE[a], J. NEBŘENSKÝ[b] and J. ULBRECHT[c]

[a]*Institute of Chemical Process Fundamentals, Czechoslovak Academy of Sciences, Prague-Suchdol*
[b]*University of Swansea, Chem. Eng. Dept., Swansea, U.K.,*
[c]*University of Salford, Chem. Eng. Dept., Salford, U.K.*

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Rheological similarity is introduced for flowing incompressible isotropic continuum and on this basis are discussed some aspects of hydrodynamic similarity of non-Newtonian flows.

If we are interested in non-Newtonian flows the question of scaling-up and modelling becomes far more complicated than in the case of hydrodynamics of Newtonian liquids. New quality becomes the choice of model liquids, which are frequently used instead of the required material not only for obtaining the dynamic similarity, but also because of reasons of health protection, security *etc.* The choice of model liquids is rather complicated by the fact that we usually do not know in advance all aspects of mechanical behaviour of individual non-Newtonian materials and are thus not able to formulate fully in advance our requirements on the model liquid. The dimensional analysis and the similarity method seem to have an open field in engineering of non-Newtonian liquids as by them even on basis of an incomplete information certain conclusions can be formulated. Their wider use was until recently prevented by lack of sufficiently adaptable way of formal description of rheological properties enabling a suitable mathematical modelling of the flow. Most of the dimensional and similarity methods were therefore until recently limited to the simplest, *i.e.* visco-inelastic rheological phenomena[1–5] which very often failed in confrontation with experiments on materials of various provenience. Only the results of non-linear mechanics of continuum attained in the last thirty years and summarized for example in the monographies by Eringen[6], and especially by Truesdell and Noll[7] led to further progress in application of dimensional resp. similarity methods in description of the flow of rheologically complex materials[8–10] as well as to some more generalized considerations concerning the hydrodynamic similarity of non-Newtonian flows[11,12].

In this study we have made an effort for a more general and systematic explanation of basical similarity problems of non-Newtonian flows. We begin with the conception of hydrodynamic similarity in the sense of identity corresponding to normalized mathematical flow[13–15] models with the conception of operational parameters defined on basis of boundary conditions and macroscopic balances[11,13] and with rheological similarity[16–19] based on the principle of dimensional invariance of rheological constitutive equations.

Mathematical Flow Model

Mathematical description of the flow based on the mechanics of continuum usually operates with the velocity field $\mathbf{v}(\mathbf{r}, t)$, the pressure field $p(\mathbf{r}, t)$ and the field of shear stress $\tau(\mathbf{r}, t)$. Without regard to the type of liquid and the flow situation, the fields of these physical quantities must for each $\mathbf{r}$, t of the limited time-space region comply with relations*

$$\varrho(\partial \mathbf{v}/\partial t + \mathbf{v} \cdot \nabla \mathbf{v}) = - \nabla(p + G) + \nabla \cdot \boldsymbol{\tau}\,, \qquad (1)$$

which means the Newton and Cauchy motion principles and with relation

$$\nabla \cdot \mathbf{v} = 0\,, \qquad (2)$$

the principle of conservation of mass for the flowing continuum.

Mathematical model of the flow is a system of relations which within the hypothesis of continuum express the mechanical aspects of flow of given liquid in a given flow situation. Apart from basical principles of conservation (*1*) (*2*), (eventually of further ones, for inst. of the principle of conservation of internal energy, enthalpy *etc.*), such mathematical flow model includes also constitutive relations which describe mathematically the relevant properties of the liquid and complementary conditions, *i.e.* boundary and initial conditions, symmetry conditions, eventually the informations on macroscopic balances which by stating certain properties of quantities $\mathbf{v}$ resp. $\boldsymbol{\tau}$ in some regions limit the type of the flow situation.

Complementary Conditions

In mathematical formulation of an actual flow problem the velocity and stress fields are not known, in general; their determination is usually the aim of solution of the mathematical model. However, some of properties of these fields are considered known or given and these constitute an important component in formulation of the mathematical model.

For our further considerations is not substantial the detailed structure of various possible types of complementary conditions, it suffices to realize several general factors which are always taken into consideration in their formulation: *1.* complementary conditions are defined in certain geometrical points (points, curves, planes, volumes limited by planes) which can be from the dimensional point of view characterized by a characteristic lengths R_c (the exceptions are *e.g.* the problems with a point momentum source, *i.e.* hydrodynamics of a jet in semiinfinite space *etc.*, whose formulation does not include any characteristic lengths). Further geometrical informations can be summarized into conditions of geometrical similarity. 2. Comple-

* All our further considerations are limited to the case of liquids with constant density ϱ and to isothermal flow. Symbol G in Eq. (*1*) represents the potential of volumetric forces.

mentary conditions which for macroscopic balances include either the characteristic kinematic information – velocity U_c, angular velocity ω_c, volumetric flow-rate Q_c *etc.* – or the characteristic dynamic criterion – force F_c, moment M_c, shear stress resp. pressure difference P_c *etc.* If the kinematic criterion is given, the dynamic parameter is the dependent quantity and *vice versa*. *3.* At the given parameter R_c the kinematic parameter can be always represented by the characteristic velocity U_c and the dynamic criterion by the characteristic pressure P_c, as for some of typical cases will be shown in Part III of this series[30]. *4.* Each hydrodynamic problem except artificially formulated ones is thus characterized from the dimensional point of view by three parameters – characteristic lengths R_c, characteristic velocity U_c, and characteristic pressure P_c. In theformulation of the problem one of the parameters P_c, U_c, R_c, is missing and its determination is the subject of solution of the mathematical model:

$$f(P_c, R_c, U_c) = 0 . \tag{3}$$

5. The three mentioned operational parameters suffice for normalisation of all complementary conditions into the dimensionless form, *i.e.* invariable in respect to change of the reference system of physical units. Since the identity of normalized complementary conditions is an indispensable condition for identity of normalized mathematical models as a whole, the three operational parameters whose values are derived from formulation of complementary conditions, are a suitable normalization mean of all quantities which appear in the formulation of the problem. *6.* The time variable can be in principle normalized by the parameter (R_c/U_c). For several types of non-stationary problems the boundary conditions include explicitly the functions of time so that on their bases can be formulated the non-stationarity of the process characterizing the time parameter t_c, which is independent on (R_c/U_c). In these problems which we shall denote as *substantially non-stationary* it is advisable to normalize the time variable as well as all other derivations in relation to the time so that the non-stationary complementary conditions in the normalized form will not contain the dimensionless parameter $St = t_c U_c / R_c$. In the problems which are not substantially non-stationary but in which, for reasons following from the time-dependence of rheological properties of considered liquids, the scale of convective time changes is nevertheless significant, it is possible to choose t_c as an arbitrary function of parameter R_c/U_c, according to the physical nature of the problem. On the contrary, there exist such substantially non-stationary problems, for which it is not possible to formulate the characteristic velocity in some other way than on the basis of the characteristic time-interval and the characteristic length (forced oscillations), *i.e.* as a function of ratio R_c/t_c. From the dimensional point of view, however, at preserving the similarity of boundary conditions, *i.e.* especially St = idem, the problem is characterized by a single parameter with time dimension s_c, which can be either t_c, R_c/U_c or their arbitrary combination.

Hydrodynamic Similarity

The complementary conditions are defined in certain geometrical locations. In solving the practical questions related especially to the model experiments resp. to the use of model liquids in these experiments, we are interested what additionally beside the geometrical similarity of these geometrical points as a whole, must be in the experiments fulfilled so that the studied phenomena are in a defined way comparable with those which can be expected from the designed unit.

The solution of these questions is usually based on an intuitively understood similarity as a generalization of geometrical similarity and the term of hydrodynamic similarity[13,14] of two processes is introduced as the identity of velocity fields, stress fields *etc.* in a suitable chosen time–space reference systems (mutually transformable by the Galilei–Newton transformations[14]) with the exception of the scale or units in which are individual quantities expressed. This means that in suitable chosen reference systems, the normalized fields of velocity, pressure and shear stresses defined by relations $t^* = t/t_c$, $\mathbf{r}^* = \mathbf{r}/R_c$, $\mathbf{v}^* = \mathbf{v}/U_c$, $p^* = p/P_c$, $\tau^* = \tau/P_c$ are identical, *i.e.* $\mathbf{v}^*(\mathbf{r}^*, t^*)$ = idem *etc.*

If two hydrodynamic situations are similar in the mentioned sense, in corresponding points and time intervals (according to Galilei–Newton transformation inclusive the change of length and time scales) for inst. velocity vectors have the same direction (with regard to geometrical objects which are a part of the considered situation) and the ratio of all pairs of in this way mutually related velocity vectors is constant and equals to the ratio of characteristic velocities *etc.* The above required "comparability" of processes in hydrodynamically similar flow situations is in this way made in a single and easily followed manner.

The mentioned formulation of hydrodynamic similarity includes implicitly some of conditions necessary for attaining it and these are the similarity conditions of complementary conditions. If the normalized fields of quantities describing the processes in two hydrodynamically similar flow situations are identical, then obviously must be identical as well the normalized expression of their properties in boundary areas, which are formulated by complementary conditions. Similarity of complementary conditions is not, however, sufficient for obtaining the hydrodynamic similarity; its further conditions are the mechanical properties of compared liquids expressed by the data of density and constitutive relations. As it is obvious that for the identity of normalized fields of velocity, stress *etc.*, the sufficient condition is the identity of normalized mathematical flow models we are looking for conditions ensuring the identity of normalized flow models instead for sufficient conditions of hydrodynamic similarity.

For example for incompressible Newtonian liquid the rheological constitutive equation $\tau = \mu\mathbf{D}$, is valid where $\mathbf{D}$ is the rate of deformation tensor and μ is the material constant, *i.e.* viscosity. The momentum balance (*1*) can be in such case modified by substituting the mentioned constitutive relation and by using the continuity Eq. (*2*) into relation

$$\varrho(\partial\mathbf{v}/\varrho t + \mathbf{v}\,.\,\nabla\mathbf{v}) = -\,\nabla(p + G) + \mu(\nabla^2)\,.\,\mathbf{v}\,,$$

in which are implicitly included all relevant mechanical properties of the incompressible Newtonian liquid at isothermal flow. In such case, the normalized momentun balance can be written in the form

$$(1/\mathrm{St})\,(\partial\mathbf{v}^*/\partial t^*) + \mathbf{v}^*\,.\,\nabla\mathbf{v}^* = \mathrm{Eu}\,\nabla^* P^* + (1/\mathrm{Re})\,(\mathbf{v}^{*2})\,\mathbf{v}^*\,,$$

where $St = t_c \cdot U_c/R_c$, $Eu = P_c/(\varrho U_c^2)$, $Re = U_c R_c \varrho/\mu$ are the criteria of dynamic similarity and $P^* = (p + G)/P_c$ is the normalized hydrodynamic potential. The necessary conditions of hydrodynamic similarity can be in this case formulated as a summary of similarity of complementary conditions and of identity of two of three dimensionless similarity criteria, for inst. St = idem, Re = idem resp. Eu = idem (according to the above given considerations identity of the third of these dimensionless parameters is already guaranted.

Rheological Similarity

For non-Newtonian liquids is the question of sufficient conditions of hydrodynamic similarity closely related with the structure of corresponding constitutive relations. Generally considered, all types of rheological constitutive relations which proved suitable in description of flow properties of isotropic incompressible materials are functional relations related to a material point (of equalities resp. inequalities) between some of tensors of relative deformations $\mathbf{d}$, tensor of shear stress $\boldsymbol{\tau}$, material time derivatives of these tensors and their history in a given material point[6,7]. In this sense, the most general constitutive equation can be formally written as

$$\mathop{H}_{s=0}^{\infty}\left[\boldsymbol{\tau}(s), \ldots, \boldsymbol{\tau}^{(i)}(s), \ldots, \mathbf{d}(s), \ldots, \mathbf{d}^{(j)}(s), \ldots, s\right] = 0\,, \tag{4}$$

where s is a time variable measuring the time from now on $s = 0$, into the past, $s \to \infty$, H is the symbol of the material functional. Time derivatives symbolized by upper indices with quantities in relation (4) are defined by relations

$$\frac{\mathrm{D}\mathbf{X}}{\mathrm{D}t} = \frac{\partial \mathbf{X}}{\partial t} + (\mathbf{v} \cdot \nabla)\,\mathbf{X} - a(\boldsymbol{\Omega} \times \mathbf{X} - \mathbf{X} \times \boldsymbol{\Omega}) + b\mathbf{D} \cdot \mathbf{X}\,, \tag{5a}$$

$$\mathbf{X}^{(k)} = \mathrm{D}\mathbf{X}^{(k-1)}/\mathrm{D}t\,; \quad \mathbf{X}^{(0)} = \mathbf{X}\,, \tag{5b),(5c}$$

where a and b are numerical constants corresponding to the type of material derivative[6,30], $\mathbf{D}$ is the rate of deformation tensor

$$\mathbf{D} = \nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{T}}\,, \tag{6}$$

for which is valid $\mathbf{D} = \mathbf{d}^{(1)}$, and $\boldsymbol{\Omega}$ is the vorticity vector

$$\boldsymbol{\Omega} = \nabla \times \mathbf{v}\,. \tag{7}$$

Limitations regarding the possible forms of the functional H, the tensor of relative deformations and material differentiations with respects to time which follow from requirements of objectivity and material indifferences[7] are further irrevelant.

Let us study the structure of the constitutive functional H (*4*) from the dimensional point of view. We see that at most by introduction of two dimensional parameters, *e.g.* of the parameter S_I with the dimension of time and of the parameter τ_I with the dimension of stress, we can transform the functional H into a dimensionless form[11]. We can obtain the same information as from the functional in the dimensional form (*4*) from values of two dimensional material constants S_I and τ_I, and the functional in the dimensionless form is

$$\overset{\infty}{\underset{s^+=0}{H^+}} \left[\tau^+(s^+), \ldots, \tau^{+(i)}(s^+), \ldots, \mathbf{d}^+(s^+), \ldots, \mathbf{d}^{+(j)}(s^+), \ldots, s^+\right] = 0\,, \qquad (8)$$

where

$$\tau^{+(i)} = \tau^{(i)}(s_I)^i/\tau_I\,, \quad \mathbf{d}^{+(j)} = \mathbf{d}^{(j)}(s_I)^j\,, \quad s^+ = s/s_I\,. \qquad (9a, b, c)$$

Complete mathematical model of the flow includes quantities which appear both in differential balances (*1*), (*2*) and in the constitutive equation (*4*). In normalization of this mathematical model there arises a question when should these quantities be normalized by operational parameters U_c, R_c, P_c resp. t_c, and when they should be normalized by material constants ϱ, τ_I, and s_I.

As some terms in differential balances can be for different liquids, at the same complementary conditions, sometimes negligible the mathematical model of the flow can be normalized so that there appear in the normalized formulation of complementary conditions as few numerical parameters-criteria of similarity as possible. This was for example the reason for introduction of a normalized time variable t^* in the form t/t_c, because with such modification the criterion St changes from the formulation of complementary conditions into the formulation of the normalized momentum balance.

As all kinematic quantities can be expressed with regard to the velocity field $\mathbf{v}(\mathbf{r}, t)$, they are normalized by use of kinematic operational parameters U_c, R_c, t_c or s_c in a form

$$\mathbf{d}^{*(j)} = \mathbf{d}^{(j)}(s_c)^j\,, \quad s^* = s/s_c\,. \qquad (10a, b)$$

The situation is different with the stress tensor. In formulation of hydrodynamic problems in the normal way, *i.e.* by giving the boundary conditions for the velocity field, the dynamic operational parameter P_c becomes a dependent variable. Obviously, it is not suitable to normalize quantities appearing in the constitutive relation (*4*) just by this parameter. Furthermore, the stress tensor in the mathematical model is relating together the differential momentum balance with the constitutive relation, while in the solution is the stress tensor eliminated in favour of the kinematic quantities. Thus the stress tensor is first of all a mean for expressing the flow properties of the given material. *Vice versa*, the material derivative is an operator whose behaviour fully depends on the flow kinematics. Therefore, the dynamic tensors are normalized in the form

$$\tau^{*(i)} = \tau^{(i)}(s_c)^i/\tau_I \,. \tag{10c}$$

The complete normalized mathematical model, apart from the normalized complementary conditions and the continuity equation in the form $\nabla^* \,.\, \boldsymbol{v}^* = 0$, consists of the normalized momentum balance

$$\mathrm{He}\left(\frac{1}{\mathrm{St}}\frac{\partial \boldsymbol{v}^*}{\partial t^*} + \boldsymbol{v}^* \,.\, \nabla^* \boldsymbol{v}^*\right) = - \mathrm{A} \,.\, \nabla^* p^* + \nabla \,.\, \boldsymbol{\tau}^* \,, \tag{11}$$

where $\mathrm{He} = \varrho U_c^2/\tau_I$ and $\mathrm{A} = P_c/\tau_I$, and the normalized constitutive relations (*10a*), (*10b*), (*10c*) are

$$\mathop{H^+}_{s^*=0}^{\infty} \left[\tau^*(s^*/\mathrm{Wd}), \ldots, \mathrm{Wd}^i \,.\, \tau^{*(i)}(s^*/\mathrm{Wd}), \ldots, \boldsymbol{d}^*(s^*/\mathrm{Wd}), \ldots, \mathrm{Wd}^j \,.\, \varrho^{*(j)}(s^*/\mathrm{Wd}), \ldots \right.$$

$$\left. \ldots, s^*/\mathrm{Wd}\right] = 0 \,, \quad \text{where } \mathrm{Wd} = s_I/s_c . \tag{12}$$

Identities of normalized mathematical models of non-Newtonian flow and consequently the hydrodynamic similarity is attained if, apart from identity of normalized complementary conditions, are fulfilled the numerical identities St = idem, He = = idem, Wd = idem and if the considered material group fulfills the condition $H^+[\ldots] =$ idem, *i.e.* if by a suitable choice of parameters τ_I and s_I which, then appear in the mentioned numerical conditions of dynamic similarity, can be attained the identity of the functionals (*8*). Because of its importance for the study of possible modelling of the non-Newtonian flows, we denote the condition $H^+[\ldots.] =$ idem as a condition of rheological similarity.

Materials with Time-Dependent Flow Properties

Although the functional H (Eq. (*4*)) includes purely from the dimensional point of view only one single independent parameter having the dimension of time it is often advisable to choose independently two parameters with the dimension of time which are the characteristic shear rate D_I and a characteristic relaxation time t_I. This is because of the fundamental importance of viscometric experiments (*i.e.* measuring of the viscosity function) on the one hand, and relaxation experiments (measuring of various time-dependent material functions) on the other hand.

In viscometric experiments the flow kinematics is characterized by the single scalar parameter **D**. Constitutive relations (*4*) based on conception of fluidity[7], can be generally reduced for viscometric flows into the form $\boldsymbol{\tau} = \boldsymbol{f}[\boldsymbol{D}]$ and normalized into the form $\boldsymbol{\tau}^+ = \boldsymbol{f}^+[\boldsymbol{D}^+]$ with the use of parameters τ_I and D_I, *i.e.* $D^+ = D/D_I$, and $\tau^+ = \tau/\tau_I$. The mentioned tensorial function of scalar argument D can also be represented by a system of three material functions among which is up to date for technical purposes the most important one the viscosity function $\tau = D \,.\, \eta[D]$ expressing the relation between the shear stress and the shear rate in the same sense as that one known from the Newton's viscosity law. Quantities τ_I and D_I can be chosen just so as to normalize into a suitable dimensionless form the viscosity function $\tau^+ = m[p] \,.\, p$ where $m = (\tau/\tau_I)/(D/D_I)$, $p = D/D_I$.

In relaxation experiments is a fundamental factor the unsteadiness of the experiment. Corresponding viscoelastic, thixotropic *etc.* properties are described by certain relaxation functions (describing for inst. successive decrease in elastic stresses after the step change of deformation, successive decrease of apparent viscosity after the step increase of shear rate *etc.*), which have generally the form $\tau = W[t]$, where τ is stress and t is the time variable. Such relations are normally normalized by introduction of characteristic relaxation time t_I into the form $\tau^+ = w[s^+]$, where $s^+ = t/t_I$ and the stress is normalized in dependence on the course of the viscosity function, *i.e.* by $\tau^+ = \tau/\tau_I$.

Generally considered constitutive relations of the type (*4*) must in some way express the both mentioned phenomena, *i.e.* they must include at least two types of quantities having the dimension of times one of which is characterizing the viscometric affects and the other the relaxation effects. This can be formally done by normalizing $\boldsymbol{D} = \boldsymbol{d}^{(1)}$ by using the characteristic deformation velocity of material D_I, $\boldsymbol{d}^{\mp(1)} = \boldsymbol{D}^{\mp} = \boldsymbol{d}^{(1)}/D_I$ in the general constitutive relation (*4*). Other operations including parameters having the dimension of time, *i.e.* material time derivatives and integrations, are normalized by use of the relaxation time t_I, *i.e.*

$$\boldsymbol{\tau}^{\mp(i)} = \boldsymbol{\tau}^{(i)}(t_I)^i/\tau_I\,, \qquad i = 1, 2, \ldots, \tag{14c}$$

$$\boldsymbol{d}^{\mp(j)} = \boldsymbol{d}^{(j)}(t_I)^{j-1}/D_I\,, \quad j = 1, 2, \ldots, \tag{14a}$$

$$s^{\mp} = s/t_I\,. \tag{14b}$$

Then holds

$$\mathop{H^{\mp}}_{s\mp=0}^{\infty}[\boldsymbol{\tau}^{\mp}(s^{\mp}), \ldots, \boldsymbol{\tau}^{\mp(i)}(s^{\mp}), \ldots, \boldsymbol{d}^{\mp}(s^{\mp}), \ldots, \boldsymbol{d}^{\mp(j)}(s^{\mp}), \ldots, s^{\mp}] = 0\,, \tag{15}$$

where $\tau^+ = \tau^{\mp}$, $d^+ = d^{\mp}$, while the form of functionals H^+ and $H^{\mp}$ differs only for a multiple constants $\mathrm{Ve} = t_I D_I$ with the kinematic tensors and time variables.

Analogical procedure, *i.e.* distinguishing between the kinematic quantities characterizing the time change and the quantities characterizing the velocity gradient, can be applied as well in normalizing the mathematical model as a whole and use consistently two parameters of the time dimension, the characteristic velocity gradient $D_c = U_c/R_c$ and the characteristic time interval t_c regardless whether D_c and t_c are independent parameters or not. Kinematic quantities can be then normalized either by parameter D_c or t_c according to the physical meaning of the normalized quantity. Partial time variations are then normalized by the characteristic time interval t_c and convective time variations are normalized like velocity gradients, *i.e.* by the characteristic velocity gradient D_c according to

$$\frac{\mathrm{D}\mathbf{X}}{\mathrm{D}t^*} = \frac{1}{t_c} \cdot \frac{\mathrm{D}\mathbf{X}}{\mathrm{D}t} = \frac{\partial \mathbf{X}}{\partial t^*} + \mathrm{St} \cdot \{(\mathbf{v}^* \cdot \nabla^*)\,\mathbf{X} - a(\ldots) + b\mathbf{D}^* \cdot \mathbf{X}\}\,, \qquad (16)$$

where all kinematic quantities in parenthesis multiplied by dimensionless St number have dimensions of a velocity gradient and are normalized in the same way. Tensor of relative deformation and its material derivatives are usually chosen so[6] as to comply with the relation $\mathbf{d}^{(1)} = \mathbf{D}$, and in general they are given by

$$\mathbf{d}^{(j)} = \mathbf{D}^{(j-1)}\,, \quad \text{for} \quad j = 1, 2, \ldots. \qquad (17)$$

Normalized kinematic tensors in a complete mathematical model are, therefore, introduced in the form

$$\mathbf{d}^* = \mathbf{d} = \mathbf{d}^{\ddagger}\,, \qquad (18a)$$

$$\mathbf{d}^{*(j)} = \mathbf{D}^{*(j-1)} = \mathbf{D}^{(j-1)} \cdot (t_c)^{j-1}/D_c\,, \quad j = 1, 2, \ldots,$$

$$s^* = s/t_c\,. \qquad (18b)$$

In the terms of so introduced dimensionless quantities of the normalized mathematical model, the functional (15) can be written in the form

$$\mathop{H}_{S^*=0}^{\infty} \left[\tau^*(s^*/\mathrm{De}), \ldots, \mathrm{De}^i \cdot \tau^{*(i)}(s^*/\mathrm{De}), \ldots, \mathbf{d}^*(s^*/\mathrm{De}), \ldots\right.$$

$$\left.\ldots, \mathrm{De}^{j-1} \cdot \mathrm{B} \cdot \mathbf{D}^{*(j-1)}(s^*/\mathrm{De}), \ldots, s^*/\mathrm{De}\right] = \mathrm{O}\,, \qquad (19)$$

where $\mathrm{De} = t_I/t_c$, $\mathrm{B} = D_c/D_I$.

From the point of view of the dimensional analysis, are thus at our disposal four parameters having the dimension of time, resp. the reciprocal time which normalize the constitutive description of the liquid and the complementary conditions. From these a number of dimensionless criteria can be formed having in spite of their apparently kinematic character the significance of criteria of dynamic similarity.

The summary of some of them is given in Table I.

The dynamic character of these criteria, which cannot be of course studied only by methods of dimensional analysis, is given by their significance in the normalized differential momentum balance and in the constitutive relation. As an example is the character of Strouhal number which is apparently only a criterion of kinematic similarity of complementary conditions. From the momentum balance (*11*) is, however, obvious that it is as well a measure of non-stationarity of the velocity fields in the Eulerian way on the overall effect of inertia forces in comparison with inertia effects arising from purely convective velocity change of mass elements of the liquid. It has the same significance in the normalized constitutive relation (*12*) which follows from definition of material derivatives in the normalized form (*16*) and from relation Wi = St De. So in the case of flows without the effect of inertia forces, St is a measure of the effect of non-stationary time changes of the velocity field on the relaxation phenomena as compared with effects of convective time changes.

Isotropic Dispersions

A number of materials can be used in rheometric experiments giving reproducible results, though dependent on the size of the apparatus, but there is an expressive heterogeneity observable by current means and often even comparable (as to the characteristic length dimension) with the dimensions of the apparatus. The hypothesis

Table I

Summary of Dimensionless Criteria with Characteristic Parameter Having the Dimension of Time

Criterion	Use
Wd $= s_1/s_c$	Criterion of dynamic similarity in the dimensionless description of the non-Newtonian flow dynamics in consequence of non-linearity (resp. non-automorfity[11,18,30]) of rheological constitutive relations
B $= D_c/D_1$	Criterion used in study of the liquid flows with non-linear or non-automorfic[11,18,30] viscosity function
Wi $= D_c t_1$	Weissenberg number used in study of visco-elastic flows in flow situations where it is not appropriate to introduce the parameter t_c. Measure of relaxation effects in steady flow situations.
De $= t_1/t_c$	Deborah number used as the measure of relaxation effects in situations where the characteristic time interval t_c can be easily defined
X $= D_1 t_c$	Not used so far
Ve $= D_1 t_1$	Viscoelastic number. Ve = idem is a part of conditions of rheological similarity
St $= D_c t_c$	Strouhal number is a part of formulation of similarity conditions of the complementary conditions

of continuum in its common meaning cannot be usually used here, and the effort to describe such systems as multiphase systems is frequent. This is usually based on the dynamics of motion of dispersed particles in the continuum media[23,24] but also alternative procedures proved good[21,22] based on phenomenological methods of the mechanics of continuum when certain phenomenological assumptions are introduced on dynamic properties of the interface boundary between the wall of the system and the bulk phase. The slip function is introduced which instead of currently used conditions of perfect adhesion of the bulk phase to the walls, expresses as a constitutive relation the dependence of the slip velocity of the bulk phase on shear stress at interfacial boundary, for inst. in a scallar form

$$v_{\mathrm{slip}} = \mathrm{f}[\tau] . \qquad (20)$$

In the description of flow properties of the bulk phase there appears an additional material parameter v_I of the velocity dimension. As well as the multiphase approach, this mentioned one based on the hypothesis of continuum introduces into the model the parameter having the dimension of length given by $r_I = v_I s_I$. While the constitutive theory including only parameters τ_I, and s_I admits scaling-up at least for creeping flows thus satisfying the modelling law D_c = const., St = idem, at the modelling conditions $R_c/(s_I v_I)$ = idem modelling with the use of original liquid is impossible.

CONCLUSIONS

In comparison with the discussed problems it is necessary in engineering practice at work with definite materials to consider only those of their flow properties which can be measured in a suitable way and not all those that are assumed by constitutive theories. Besides, the present situation in engineering of non-Newtonian liquids, especially of visco-elastic materials, rough and thixotropic suspensions, is furthermore characterized by lack of suitable rheometric experiments which have been until now limited overwhelmingly to experiments in visco-metric flow situation, experiments of "linear visco-elasticity", or technological experiments with undefinable flow kinematics.

This is at best reflected in experiments in which some flow situation with visco-elastic liquids were correlated based on conception of material relaxation time t_I. Bird[5] based his work directly on visco-metric experiments only, other engineering-oriented authors define the relaxation time t_I on basis of measurements of the viscosity function and of differences of normal stresses[25,26] at various viscometric flow situations. Though the latter attitude led to satisfactory correlation for some flow situations interesting from the engineering point of view with the use of Deborrah and Weissenberg numbers[25-27] and it demonstrates the relation between the relaxation phenomena and the existence of normal pressure differences, but its use is obviously limited to a narrow class of model liquids since new rheometric experiments[28,29] present now already quantitatively, further phenomena which cannot be correlated with the results of experiments in only visco-metric flow situations.

It is useful to note that any experimentally determined flow properties are expressed *de facto* as dynamic characteristics, *i.e.* dependences of the type $P_c = f(U_c, R_c, t_c)$ between the operational parameters characterizing the activity of a given rheometrical apparatus. In the engineering analysis comprising the use of model liquids can be, instead of *a priori* supposed constitutive relations often evaluated on basis of a small number of unsuitably chosen rheometric experiments[5], mutually correlated the dynamic characteristics of rheometric experiments whose analysis is otherwise difficult on the one hand and dynamic, kinematic and transport characteristics obtained by model experiments in technically-interesting flow situation on the other hand. So just there where in the analysis of results of rheometric experiment we cannot start either from an exact hydrodynamic analysis independent of the type of supposed constitutive relation, or can be supposed such suitable relation in the problem of correlation relation, the dimensional and similarity correlation methods will always have their place.

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LIST OF SYMBOLS

$\mathbf{d}$	tensor of relative final deformations
$\mathbf{d}^{(j)}$	kinematic tensor of the j-th order (s^{-j})
$\mathbf{D}$	the rate of deformation tensor (s^{-1})
D	deformation rate, second invariant of $\mathbf{D}$ (s^{-1})
D_I	material characteristic deformation rate (s^{-1})
$D_c = U_c/R_c$	characteristic velocity gradient of flow situation (s^{-1})
$\mathbf{f}$	tensor function
G	potential of volumetric forces (cm s^{-1})
H	material functional, constitutive rheological relation
H^+, $H^{\ddagger}$	normalized material functions
p	isotropic pressure (dyn cm^{-2})
p^*	normalized hydrodynamic potential
P_c	characteristic stress resp. pressure difference (dyn cm^{-1})
$\mathbf{r}$	position vector (cm)
r_I	characteristic dimension of heterogeneity (*e.g.* particle dimension) of the disperse medium (cm)
R_c	characteristic dimension of the given flow situation (cm)
s	time variable in the material functional (s)
s_I	material constant (s)
s_c	parameter of complementary conditions (s)
t	time (s)
t_I	material relaxation time (s)

t_c characteristic time interval of the flow situation, characterizing the time changes in the sense of substantial time derivative (s)
U_c characteristic velocity of flow situation (cm s^{-1})
V_I characteristic slip velocity, material constant (cm s^{-1})
$\mathbf{v}$ liquid velocity (cm s^{-1})
$\mathbf{X}$ arbitrary symmetric tensor of second order

Dimensionless Numbers

$A = P_c/\tau_I$ parameter, supplying in Eq. (*11*) the Euler number
$B = D_c/D_I$ dimensionless number, see Table II
$De = t_I/t_c$ Deborah number
$Eu = P_c/\varrho U^2$ Euler number
$He = \varrho U_c^2/\tau_I$ Hedström number, representing in Eq. (*11*) the Reynolds number
$Re = \varrho U_c R_c/\mu$ Reynolds number for non-Newtonian liquids
$St = D_c t_c$ Strouhal number
$Ve = D_I t_I$ viscoelastic number
$Wi = t_I D_c$ Weissenberg number
$Wd = s_I/s_c$ Wilde number

Greek symbols

∇ differential operator nabla (cm^{-1})
∇^2 Laplace differential operator (cm^{-2})
η apparent viscosity, viscosity function (P)
μ Newtonian viscosity (P)
ϱ density (g cm^{-3})
$\boldsymbol{\tau}$ tensor of deformation stress (dyn cm^{-2})
τ deformation stress (dyn cm^{-2})
τ deformation stress, second invariant of tensor $\boldsymbol{\tau}$ (dyn cm^{-2})
τ_I characteristic stress, material constant (dyn cm^{-2})

Superscripts

T transposition of second order tensor $(x_{ij})^T = x_{ji}$
$+$ quantities, normalized by parameters τ_I, s_I
$\ddagger$ quantities, normalized by parameters τ_I, t_I, D_I
$*$ quantities, normalized by operating parameters R_c, t_c resp. s_c
(i), (j) i-th resp. j-th material derivative, see Eq. (*5*)

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